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Graph splicing systems☆

Rahul Santhanam, Kamala Krithivasan*

Department of Computer Science and Engineering, Indian Institute of Technology, Madras, 600 036, India

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Abstract

In this paper, extended graph splicing systems are defined. It is shown that when strings are represented as linear graphs, any recursively enumerable set can be generated by an extended graph splicing system. It is also shown that the computational completeness of extended graph splicing systems can be proved under some constraints too.

Keywords: Extended graph splicing systems; Computational completeness; Constrained splicing systems

1. Introduction

Splicing systems were introduced by Head in [5] to investigate the power of computing with DNA [8]. Some decidability results about splicing systems were studied in [2]. One of the recent developments in this field is the generalization of the concept of "splicing system" to graphs by Rudolf Freund in [3]. Freund's formulation is interesting because it is quite an accurate model of a system consisting of biochemical units and the interactions between them. In this paper, we present results concerning the generative capacity of certain versions of the splicing system defined in [3]. We prove that, if a distinction between terminals and nonterminals is made in the system, then these graph splicing systems with finite sets of axioms and rules are computationally complete in a certain sense. If we consider a graph language consisting only of directed paths to be equivalent to the corresponding string language, then these systems can generate all recursively enumerable sets of strings.

Graph splicing systems have various parameters like number of axioms, number of rules etc. We examine, in this paper, the effect of constraints upon these parameters on the computational completeness of splicing systems. We find the minimum values of these parameters for which the resulting (constrained) class of systems is still computationally complete. These results have practical significance since the values of the parameters will be bounded in practical biochemical situations. The completeness of the splicing systems even for small bounds on the parameters provides a good indication for the feasibility of practical bio-computers.

The layout of the paper is as follows. In Sections 2 and 3 we give a few concepts from the literature which are required for the understanding of the paper. In Section 4, we prove the computational completeness of graph splicing systems. In Section 5, we study the effect of imposing some constraints on graph splicing systems in terms of computational power. We conclude the paper in Section 6.

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* Corresponding author.
E-mail address: kamala@iitm.ernet.in (K. Krithivasan).

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2. Preliminaries

Let Σ be any finite alphabet. Σ^* denotes the set of all strings over Σ. A graph G over Σ is a triple (V, E, f) where V is the node set, $E \subseteq V \times V$ is the edge set and f is the vertex labelling function ($f : V \to \Sigma$). Given a graph G, $V(G)$ denotes the vertex set of G and $E(G)$ the edge set of G. A path in G between vertices v_0 and v_n belonging to $V(G)$ is a set of vertices $v_0, v_1, \ldots, v_n$ such that there is an edge between v_i and v_{i+1} for $i = 0, \ldots, n-1$. A graph G is said to be connected if there is a path in G between every pair of vertices that belong to $V(G)$. A graph language is a set of labelled graphs.

The graphs, $G_1 = (V_1, E_1, f_1)$ and $G_2 = (V_2, E_2, f_2)$, are said to be equivalent if there is a bijective function $g : V_1 \to V_2$ such that,

- for $x, y \in V_1$, $(x, y) \in E_1$ iff $(g(x), g(y)) \in E_2$,
- for all $x \in V_1$, $f_1(x) = f_2(g(x))$.

The above relation induces an equivalence relation creating equivalence classes on the set U of all graphs. Each such equivalence class is called an *abstract graph*. The abstract graph that corresponds to a graph G is represented by $[G]$. When we consider graph languages, we mean languages of abstract graphs. When we deal with unlabelled graphs, this notion is called Isomorphism.

We now define the notion of a Turing machine [6]. A Turing machine is defined by the tuple $M=(Q, \Sigma, \Gamma, \delta, q_0, B, F)$ where Q is the set of states, Σ is the input alphabet, Γ is the set of tape symbols, δ is a mapping from $Q \times \Gamma$ to $Q \times \Gamma \times \{L, R\}$, q_0 is the initial state, B is a symbol in Γ called the blank symbol and F is the set of final states.

We denote an *instantaneous description* (ID) of the Turing machine by $\alpha_1 q \alpha_2$. Here q, the current state of M, is in Q; $\alpha_1\alpha_2$ is the string in Γ^* that is the contents of the tape. The tape head is assumed to be scanning the leftmost symbol of α_2, or if $\alpha_2 = \varepsilon$, the head is scanning a blank symbol. We denote the moves of the Turing machine by the relation $\vdash$ defined on the IDs of the machine. The relation, $\vdash^*$, denotes the reflexive transitive closure of $\vdash$. We refer the reader to [6] for further details. The language accepted by a Turing machine, $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$, denoted by $L(M)$, is defined as

$$L(M) = \{w | w \in \Sigma^* \text{ and } q_0 w \vdash^* \alpha_1 p \alpha_2 \text{ for some } p \text{ in } F, \text{ and } \alpha_1 \text{ and } \alpha_2 \text{ in } \Gamma^*\}.$$

3. Definitions

Definition 1 (*Graph splicing system*). A graph splicing system σ is a pair (Σ, P) where Σ is an alphabet and P is a finite set of rules of the form, $((h[1], E'[1]), \ldots, (h[k], E'[k]); E)$, where

- $h[i] = (V[i], E[i], L[i])$, $1 \leqslant i \leqslant k$, is a graph ($L[i]$ is the vertex labelling function),
- $E'[i] \subseteq E[i]$, $1 \leqslant i \leqslant k$. $E'[i]$ is called the set of cut-edges;
- the node sets, $V[i]$, are mutually disjoint;
- E must obey the following rules:
  - Each edge $(n, m) \in E'[i]$ is supposed to have been divided into two parts, i.e. the start part $(n, m]$ and the end part $[n, m)$ (refer to Fig. 1).
  - The elements of E are pairs of the form $((n, m], [n', m'))$, where (n, m) and (n', m') are edges from $\bigcup E'[i]$.
  - Every element from $\{((n, m], [n, m)) | (n, m) \in \bigcup E'[i]\}$ must appear exactly once in a pair of E. □

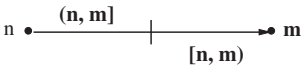

Fig. 1. View of a cut-edge.

Definition 2 (*Graph splicing step*). Let $\sigma = (\Sigma, P)$ be a graph splicing system. Let $p = ((h[1], E'[1]), \ldots, (h[k], E'[k]); E)$ be a rule in P. We say that a set R of graphs derives a set S of graphs through the splicing rule p if there exist graphs $g[1], g[2], \ldots, g[k] \in R$ and graphs $g'[1], g'[2], \ldots, g'[m] \in S$ such that,

- For all i with $1 \leqslant i \leqslant k$, $h[i]$ is an induced subgraph[1] of $g[i]$.
- From $g[1], g[2], \ldots, g[k]$ we delete all edges corresponding to edges in $\bigcup E'[i]$ but add each edge corresponding to the edge (n, m') such that $((n, m][n', m')) \in E$, which yields the uniquely determined union of m connected graphs $g'[1], g'[2], \ldots, g'[m]$. □

Here we are implicitly using multisets of graphs. $g[j]$ may be another copy of $g[i]$. The basic idea of a graph splicing rule is to cut various edges in some connected graphs and rejoin the edges in another fashion.

We give below the definition of an extended graph splicing system. It is slightly different from the definition given in [3].

Definition 3 (*Extended graph splicing system*). Let $\sigma = (\Sigma, P)$ be a graph splicing system and I be a finite set of graphs, called the set of axioms. The quadruple, (N, T, P, I) where N and T are disjoint sets of nonterminals and terminals, respectively, and $N \cup T = \Sigma$, is called an *extended graph splicing system* (in an ordinary graph splicing system, no distinction between terminals and nonterminals is made). $\sigma(I)$ is the minimal set of graphs obtained by applying some splicing rules to some subset of I. $\sigma^n(I)$ is defined iteratively as $\sigma(\sigma^{n-1}(I))$; $\sigma^0(I)$ is defined to be I. We also define $\sigma^*(I) = \bigcup_{n \geqslant 0} \sigma^n(I)$. The graph language, $L(F)$, generated by an extended graph splicing system F, $F = (N, T, P, I)$, is

$$L(F) = \{g = (V, E, f) \in \sigma^*(I) | f(n) \in T \ \forall n \in V(g)\}. \qquad \square$$

The form of the splicing rules in a graph splicing system places a restriction on the structure of the corresponding graph language. We define a graph language L to be of *bounded degree* if there is an integer k such that no vertex of any graph in L has degree greater than k. For example, the set of all binary trees is a graph language of bounded degree while the set of all complete graphs is not. All graph splicing languages are graph languages of bounded degree. This is because the degrees of vertices of graphs participating in splicing are preserved in a graph splicing step. Thus, if $S = (N, T, P, I)$ is an extended graph splicing system then,

$$\text{Maximum degree of any vertex of a graph in} L(S) \leqslant \max\{\text{degree of } v | v \text{ is a vertex of a graph in } I\}.$$

Since the axiom set I is finite, the maximum exists and is an integer that bounds the degree of any vertex of a graph in L.

A graph language is said to be *Turing machine representable* if the set of effective representations of the graphs in the language is recursively enumerable (r.e.). Here "effective representation" means a representation that can be generated from the standard set representation of a graph by a Turing machine. The adjacency matrix of a graph, if considered in a linearized (row-major or column-major) form, is an example of such a representation. It is obvious that there are graph languages that are Turing machine representable but not degree-bounded, for example the set of complete graphs. A Turing machine could easily be designed that accepts all sequences of n^2 1's for integer n and such a sequence is an effective representation of a complete graph. Thus the class of Turing machine representable graph languages does not coincide with the class of graph splicing languages. It is an open question whether the class of Turing machine representable graph languages of bounded degree is identical to the class of graph splicing languages.

We next consider a subclass of graph languages called *linear graph languages*. A linear graph language is a graph language that consists only of directed paths. The effective representation of a linear graph can be the string they represent. For example, the graph shown in Fig. 2 represents the string $abc \cdots n$. Example 1 gives a graph splicing system which has all its graphs to be linear.

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[1] A graph $H = (V', E')$ is an induced subgraph of a graph G, $G = (V, E)$ if,

$$\forall a, b \in V', \quad (a, b) \in E' \text{ iff } (a, b) \in E.$$

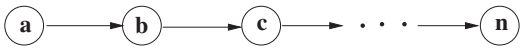


Fig. 2. A linear graph.

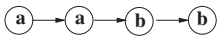


Fig. 3. The graph in the set of axioms, I.

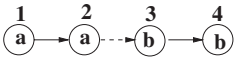


Fig. 4. The graph, $h[1]$.

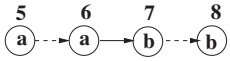


Fig. 5. The graph, $h[2]$.

Example 1. Consider the graph splicing system σ defined as, $\sigma = (N, T, P, I)$ where the set of axioms, I, is given in Fig. 3. The set P consists of the rule, $R = ((h[1], E'[1]), (h[2], E'[2]), E)$, where the graphs $h[1]$ and $h[2]$ are shown in Figs. 4 and 5, respectively. The cut-edges are shown in dotted lines in the figures.

The set E is defined as

$$E = \{(2, 6), (7, 3), (5, 8)\},$$

where the labelling is as in the Figs. 4 and 5. The language generated by the system consists of the linear graphs representing the language, $\{a^n b^n | n \geqslant 1\}$.

In the next section, we prove that for every r.e. language L, there is a graph splicing system that generates the linear graph language representing the language L.

4. A completeness result

Computational completeness of extended regular H systems is provided in [7]. In this section, we prove the computational completeness of graph splicing systems. We prove that, for every r.e. language L, there is a graph splicing system that generates the linear graph language corresponding to L. The basic idea of the proof is to simulate the working of a one-tape Turing machine M accepting L using a graph splicing system. Since every r.e. set is accepted by some one-tape Turing machine, the simulation establishes the result.

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, \{q_f\})$ be a Turing machine (without loss of generality, we can assume that there is only one final state). We construct an extended graph splicing system $F = (N, T, P, I)$ to simulate the working of M as follows. We define, $V = \Gamma \cup Q \cup \{C, D, E, X, Y, Z\}$, $N = V \backslash \Sigma$ and $T = \Sigma$. The graphs in the set I are given in the Fig. 6.

The prime idea behind the simulation of the working of the Turing machine on a string in Σ^* is as follows. For each string in Σ^*, we generate a graph with two copies of the string, i.e., a graph of the form shown in Fig. 7. Then, we simulate a Turing machine on one of the copies of the string. If a halting state is reached, the part of the graph involved in the simulation is cut off and the linear graph representation of a string belonging to the r.e. set is generated.

We now describe the rule set P. The set P consists of nine classes of rules. The classes are described in the Tables 1 and 2. Each class of rules are such that every rule in the class has exactly two graphs, $h[1]$ and $h[2]$. For the sake of brevity, we have given the graphs, $h[i]$'s and the resultant graphs obtained from these graphs after the splitting and recombining for each class of rules. We depict the cut-edges as dotted lines in the graphs $h[i]$'s.

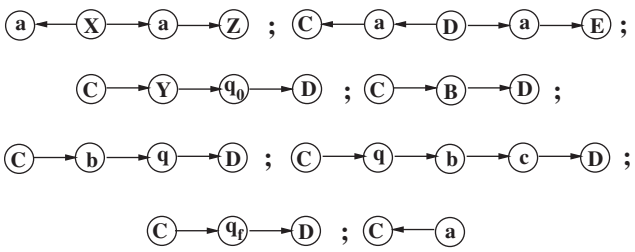


Fig. 6. The graphs in the set I. Here, $a \in \Sigma$, $b, c \in \Gamma$, $q \in Q$.

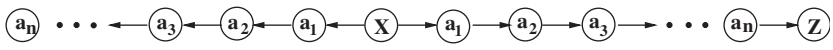
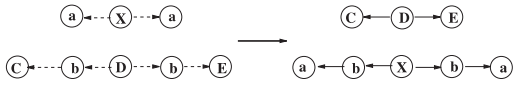


Fig. 7. Graph generated during the simulation. Here the string is $w = a_1 a_2 \cdots a_n$.

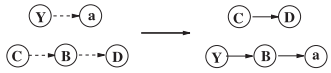
Table 1
Classes of rules in P

Class 1: In the figure, $a, b \in \Sigma$.

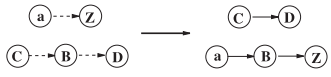


Class 2: In the figure, $a \in \Sigma$.

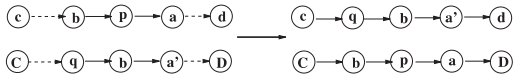
Class 3: In the figure, $a \in \Sigma$.



Class 4: In the figure, $a \in \Sigma$.



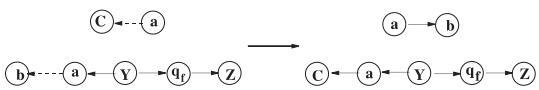
Class 5: In the figure, $a, b, a' \in \Gamma$, $c, d \in N$, $p, q \in Q$ such that $\delta(p, a) = (q, L, a')$.



Class 1 of rules is used for generation of a graph containing two copies of a string. Class 2 is used for beginning the simulation, by bringing a symbol for the start state into the graph. Classes 3 and 4 are used for introducing blanks, as and when needed. Classes 5 and 6 are used for the actual simulation, each application of a rule simulating one step of a Turing machine. Class 5 is used for the left moves of the tape head and Class 6 for the right moves. Classes 7 and 8 are used for incrementally deleting the second copy of the string from the graph if a halting state is reached. Class 9 is used for finally generating a string that has been accepted by the Turing machine.

To elaborate further, Class 1 rules generate the graph as shown in Fig. 7. Class 2 starts the simulation by introducing q_0 after X and simultaneously changing X to Y. Successive IDs of the Turing machine occur between X and Z. Rules

Table 2
Classes of rules in P

Class 6: In the figure, $a, d, a' \in \Gamma$, $b \in N$, $p, q \in Q$ such that $\delta(p, a) = (q, R, a')$. 
Class 7: In the figure, $a, c \in N$, $b \in \Gamma$. 
Class 8: In the figure, $a, c \in N$, $b \in \Gamma$. 
Class 9: In the figure, $a, b \in \Sigma$. 

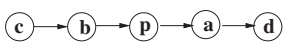


Fig. 8. Linear graph of the ID before applying the rules in Class 5.

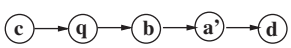


Fig. 9. Linear graph of the ID after applying the rules in Class 5.

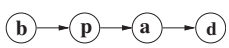


Fig. 10. Linear graph of the ID before applying the rules in Class 6.

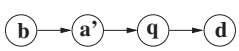


Fig. 11. Linear graph of the ID after applying the rules in Class 6.

in Class 5 simulate a left move namely, $\delta(p, a) = (q, a', L)$. The corresponding portion of the ID in the linear graph as shown in Fig. 8 is changed accordingly to obtain the linear graph representation of the resultant ID as shown in Fig. 9 by splicing using these rules. Rules in class 6 simulate a right move, $\delta(p, a) = (q, a', R)$. The corresponding portion of the ID in the linear graph as shown in Fig. 10 is changed accordingly to obtain the linear graph representation of the resultant ID as shown in Fig. 11 by splicing using these rules. By this simulation, successive IDs appear between Y and Z. Once the final state q_f is reached, symbols next to q_f are consumed by the rules in Classes 7 and 8 ending up with the linear graph shown in Fig. 12. The 9th class rules remove the Y, q_f and Z ending up with $w = a_1 a_2 \ldots a_n$ in the linear graph.

Fig. 12. Linear Graph obtained after applying rules in Classes 7 and 8.

A proof of the equivalence between the language accepted by the Turing machine and the language generated by the graph splicing system is straightforward. It can be seen that any string accepted by the Turing machine is also generated as a linear graph by the graph splicing system as described above. The graph language consists of graphs having terminal symbols as node labels produced by the system. Only strings accepted by the Turing machine will be produced by the system with terminal node labels. The form of the rules and the form of any byproducts generated during a splicing preclude the possibility of extra strings (strings not accepted by the Turing machine) being generated. Those graphs will have nonterminal node labels. Therefore, the equivalence holds.

5. Results on constrained splicing systems

Consider a class C of graph splicing systems with finite sets of axioms and rules. The family of graph languages, $\mathscr{L}(C)$, corresponding to C is defined as

$$\mathscr{L}(C) = \{L(F) | F \in C\}.$$

If $\mathscr{L}(C)$ contains the class of recursively enumerable languages represented as linear graphs, then C is said to be computationally complete. We proved earlier that the class U of all graph splicing systems with finite sets of axioms and rules is computationally complete.

Next we define the following terms:

$$\alpha(C) = \max\{|I| | F = (N, T, P, I) \in C \text{ for some } N, T, P\},$$
$$\beta(C) = \max\{|P| | F = (N, T, P, I) \in C \text{ for some } N, T, I\},$$
$$\gamma(C) = \max\{|N| | F = (N, T, P, I) \in C \text{ for some } T, P, I\},$$
$$\delta(C) = \max\{k | k \text{ graphs occur in a rule of } P \text{ for some } F = (N, T, P, I) \in C\},$$
$$\omega(C) = \max\{k | k \text{ edges are split in some rule of } P \text{ for some } F = (N, T, P, I) \in C\}.$$

Most of the results in this section are obtained by slightly modifying the simulation of a Turing machine that yielded the earlier completeness result. Therefore, instead of giving the detailed simulation for each result, we specify the changes to be made to the original simulation.

Theorem 1. *There is a set C_1 of graph splicing systems with $\alpha(C_1) = 1$ that is computationally complete.*

Proof. In order to prove the theorem, we shall show that any r.e. language represented as linear graphs can be generated by a graph splicing system with a single axiom. We mainly use the simulation techniques used in the previous section. The basic idea of the proof is to encode many axioms by one axiom, from which the original axioms can be derived by the application of a special rule.

Let L be a r.e. language. Then, by the result in the previous section, there exists an extended graph splicing system, $F = (N, T, P, I)$, which generates L (represented as linear graphs). We construct a graph splicing system $F' = (N', T, P', I')$ from F such that $L(F) = L(F')$ and $|I'| = 1$.

Let the axioms in I be $I_1, I_2, \ldots, I_n$. Then F' has $N' = N \cup \{c_1, c_2\}$ where c_1 and c_2 are symbols not in N. I' consists of a single axiom which encodes the information in all the axioms $I_1, I_2, \ldots, I_n$ of F. Let us abuse notation by referring to the single graph in I' as I'. Then, I' is given in Fig. 13. The subgraphs J_i refers to the graphs shown in Fig. 14. In the subgraph J_i, a_i is the first symbol of I_i and a_i' is the last symbol of I_i, for each i. The symbol a in I' is any symbol in N. Each J_i is attached transversely in I', i.e., each J_i is connected to its neighbours only through the node labelled c_2.

The rule set P' of F' is defined as,

$$P' = P \cup Q_1 \cup Q_2 \cup \cdots \cup Q_n,$$

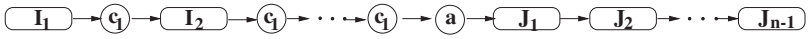


Fig. 13. The new axiom, I'. Here I_k's are the subgraphs representing the old axioms. The subgraphs J_k's are shown in Fig. 14.

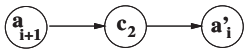


Fig. 14. The subgraph J_i, $1 \leqslant i \leqslant n-1$.

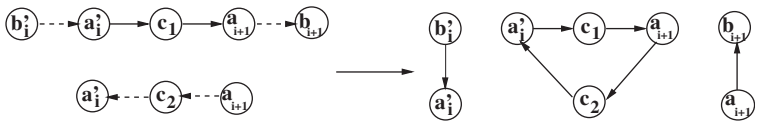


Fig. 15. The rule Q_i, $1 \leqslant i \leqslant n$.

where the rule Q_i, is as shown in Fig. 15. In the rule Q_i, b_i' is the penultimate symbol of I_i and b_i is the second symbol of I_i.

First, we prove that $L(F) \subseteq L(F')$. It is enough to prove that $I_1, I_2, \ldots, I_n$ can be derived in F'. To do this, we successively apply the rules $Q_1, Q_2, \ldots, Q_n$ to I'. When Q_1 is applied to I', I_1 and $I_2 \to \cdots$ are derived. When Q_2 is applied, I_2 and $I_3 \to \cdots$ are derived. Thus, by successively applying the rules Q_i, the axioms of F can be derived. By applying the rules in P to these axioms, all graphs that belong to $L(F)$ can be derived in F'.

The reverse inclusion, $L(F') \subseteq L(F)$, is much more difficult to prove. The proof is by induction. We show that, if a certain set of properties is true of graphs derived in n or fewer steps, it is also true of graphs derived in $n+1$ steps. We also show that this set of properties constrains graphs with only terminal nodes that are derivable in F' to also be derivable in F.

The set of properties is as below.

1. If the graph does not contain the nonterminals c_1 and c_2, it is derivable in F.
2. If the graph contains the nonterminal c_1 but not c_2, it is of the form $w_1 \to c_1 \to \cdots \to w_r \to c_1, r \geqslant 2$ where the w_i's are linear graphs derivable in F.
3. If the graph contains the nonterminals c_1 and c_2, it is of the form $w_1 \to c_1 \to \cdots \to c_1 \to a \to K_1 \cdots K_{n-1}$. Here, each K_i is either equal to J_i or it is J_i with a node labelled c_1 attached to a_i' and a_{i+1}. Also, the w_i's are linear graphs derivable in F.

We now show that the properties above are conserved in an application of a graph splicing rule.

Let the splicing rule be applied to graphs G_1 and G_2. If neither of the graphs has property 3, only rules in P can be applied and the resultant graphs will have either property 1 or property 2. If a rule in P is applied to graphs having any of the three properties, it is easily seen that one of the properties holds good for each of the resulting graphs. The only remaining case is the situation where one or both of G_1 and G_2 have property 3 and one of the rules Q_i, $1 \leqslant i \leqslant n$ is applied. If the K_i's are equal to the J_i's, we obtain one graph with either property 1 or property 2 and two graphs with property 3. Otherwise,we get the same graphs G_1 and G_2. In either case, one of the three properties holds for each of the resultant graphs.

For the case $n = 0$, the property 3 holds. Since the properties are preserved by a splicing step, one of the 3 properties holds for any graph derived in F'. We are interested only in graphs with all nodes having terminal labels. Such graphs can only have property 1, therefore they are derivable in F, i.e., $L(F') \subseteq L(F)$. □

Theorem 2. *There does not exist a computationally complete set C_3 of graph splicing systems such that $\gamma(C_3) = 0$.*

Proof. This result is given in [4]. It is obvious that languages such as the regular language $(aa)^*$ cannot be generated by a graph splicing system without nonterminals. □

Theorem 3. *There exists a set C_3 of graph splicing systems with $\gamma(C_3) = 1$ such that C_3 is computationally complete.*

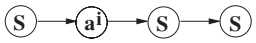


Fig. 16. Replacement subgraph for N_i.

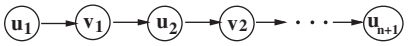


Fig. 17. The graph $h[1]$ of p'.

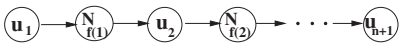


Fig. 18. The graph $h[1]$ of p.

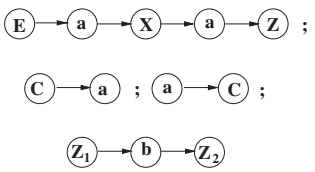
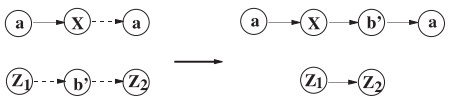


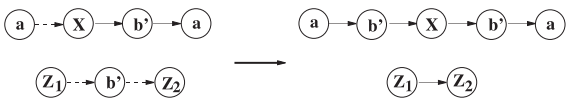
Fig. 19. Axiom set I. Here $a \in \Sigma$, and $b \in V$.

Table 3
Classes of rules in P

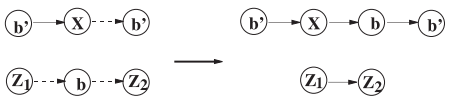
Class 1: In the figure, $a, b \in \Sigma$.



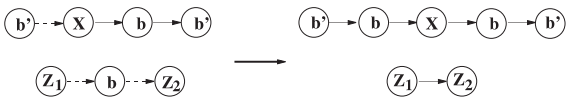
Class 2: In the figure, $a, b \in \Sigma$.



Class 3: In the figure, $b \in \Sigma$.



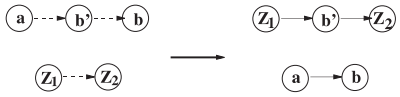
Class 4: In the figure, $b \in \Sigma$.



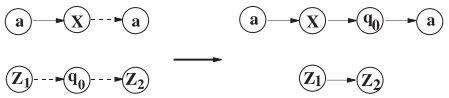
Class 5: In the figure, $a, b \in \Sigma$.

Table 4
Classes of rules in P

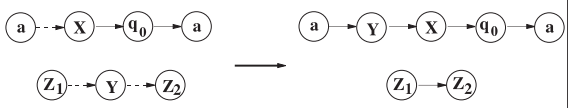
Class 6: In the figure, $b \in \Sigma$.



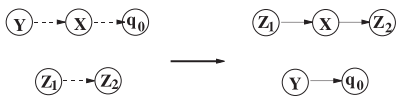
Class 7: In the figure, $a \in \Sigma$.



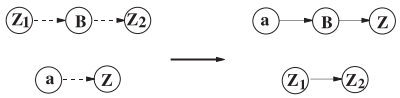
Class 8: In the figure, $b \in \Sigma$.



Class 9:



Class 10: In the figure, $a \in Q \cup \Gamma$.



Proof. We simulate a graph splicing system $F = (N, T, P, I)$ with a graph splicing system $F' = (N', T', P', I')$ which has $|N'| = 1$. The simulation is a slight modification of the standard Turing machine simulation. Consider a symbol $a \in T$. Let $|N| = k$. We code the nonterminals in N using only a and a single nonterminal S. Let $N_1, N_2 \cdots N_k$ be an enumeration of the members of N. Then N_i is coded as $Sa^i SS$. P' is the same set of rules as P except that N_i is replaced by the subgraph shown in Fig. 16, wherever it occurs. I' is the same set of axioms as I except that N_i is replaced appropriately wherever it occurs.

First, we prove that $L(F) \subseteq L(F')$. This is obvious, since axioms and rules used in the derivation of a graph in F can be replaced by corresponding axioms and rules of F' with the derivation still holding.

The proof that $L(F') \subseteq L(F)$ is slightly more difficult. The proof is by induction on the number of steps required to derive a graph in F'.

Consider the function $f : N \to \{S, a\}^*$ defined by

$$f(N_i) = Sa^i SS.$$

Extend the function to $f : (N \cup T)^* \to (\{S\} \cup T)^*$ in the usual way.

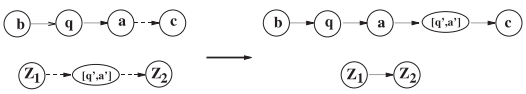
The induction hypothesis is as follows. If a linear graph w is derivable in F' in n steps or less, there is a unique graph w' corresponding to it in F that is derivable in F in the same number of steps.

The base case is $n = 0$. The graphs of F' derivable in zero steps are the axioms in I'. These obviously correspond to unique axioms of I (by construction).

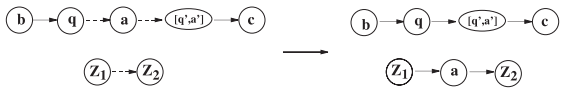
Table 5
Classes of rules in P

Class 11: In the figure, $a \in Q \cup \Gamma$.

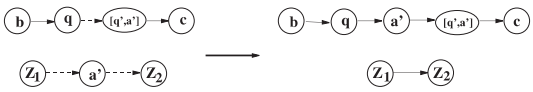
Class 12: In the figure, $b, a, c, a' \in \Gamma, q, q' \in Q, \delta(q, a) = (q', a', R)$.



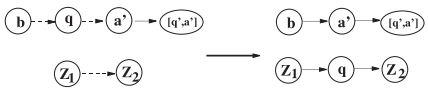
Class 13: In the figure, $b, a, c, a' \in \Gamma, q, q' \in Q$.



Class 14: In the figure, $b, a', c \in \Gamma, q, q' \in Q$.



Class 15: In the figure, $b, a' \in \Gamma, q, q' \in Q$.



Let us assume that the induction hypothesis is true for $n = n_0$. Consider a linear graph that is derivable in $n_0 + 1$ steps in F'. This implies that there are linear graphs w_1 and w_2 in F' and a rule p' of F' such that w_1 and w_2 are derivable in n_0 or fewer steps and the application of p' to w_1 and w_2 yields w.

Consider the linear graph $h[1]$ of p'. Let this graph be as given in Fig. 17. In the graph, u_i's belong to T^* and v_i's are codings of nonterminals in F. If v_i is a coding of $N_{f(i)}$ for each i, then this rule corresponds to a rule p of F which has $h[1]$ as given in Fig. 18. $h[1]$ is an induced subgraph of w_1. In w_1, the subgraph in Fig. 17 can only be a code of the graph in Fig. 18. This is because the first character of v_1 is an S and it is followed by a terminal. This can only indicate the beginning of a code for a nonterminal. Similar reasoning holds for the graph $h[2]$ of p' and w_2. Thus, if w_1 and w_2 yield w by splicing using p', then their corresponding graphs in F yield the corresponding graph of w in F by splicing using p. Thus, this graph is derivable in F in $n_0 + 1$ steps.

In particular, linear graphs with only terminal labels code for themselves. Therefore, if a string $w \in T^*$ is derivable in F', it is derivable in F. This implies $L(F') \subseteq L(F)$. □

Theorem 4. *There does not exist a set C_4 of graph splicing systems with $\delta(C_4) = 1$ that is computationally complete.*

Proof. If there is only one graph in each rule, then the size of graphs in the corresponding splicing language is bounded. This means that the splicing language is finite. Therefore, no infinite set of strings can be generated by a splicing system with maximum 1 graph per rule. □

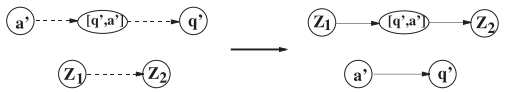
Theorem 5. *There exists a set C_5 of graph splicing systems with $\delta(C_5) = 2$ that is computationally complete.*

Proof. Graph splicing systems used in the Turing machine simulation of Section 5 are precisely of this form. □

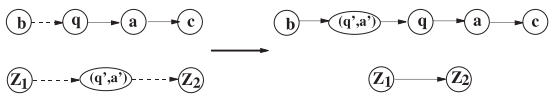
Table 6
Classes of rules in P

Class 16: In the figure, $a', c \in \Gamma, q' \in Q$.

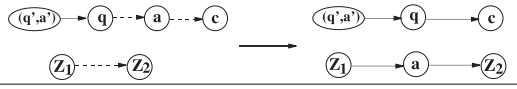
Class 17: In the figure, $q' \in Q, a' \in \Gamma$.



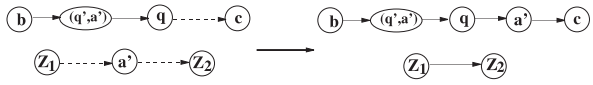
Class 18: In the figure, $b, c, a, a' \in \Gamma, q, q' \in Q, \delta(q, a) = (q', a', L)$.



Class 19: In the figure, $b, a, c, a' \in \Gamma, q, q' \in Q$.



Class 20: In the figure, $b, a', c \in \Gamma, q, q' \in Q$.



Theorem 6. *There exists a set C_6 of graph splicing systems with $\omega(C_6) = 3$ that is computationally complete.*

Proof. We prove this by actually constructing, for every Turing machine M, a splicing system in which no more than three edges are cut and which simulates the Turing machine. The splicing system generates, for each string, a graph containing a copy of the string and a copy of the reverse of the string. Then we make use of the property that, for each Turing machine M, there exists a Turing machine M^R such that $w \in L(M)$ iff $w^R \in L(M^R)$. We simulate the machine M^R on a copy of w^R and generate w iff w^R is accepted by M^R.

Let $M^R = (Q, \Sigma, \Gamma, \delta, q_0, B, \{q_f\})$. Then we construct a graph splicing system (N, T, P, I) as follows:

$$V = \Gamma \cup Q \cup \{E, X, X', Y, Z, C, Z_1, Z_2\} \cup \{[q, a] | q \in Q, a \in \Gamma\} \cup \{(q, a) | q \in Q, a \in \Gamma\} \cup \{a' | a \in \Gamma\}.$$

The axiom set I is given in Fig. 19. The rule set P has 29 finite classes of rules. We describe each class pictorially in Tables 3–7. Each rule has two graphs, $h[1]$ and $h[2]$. We describe these graphs and the resultant graphs obtained from these graphs after splitting and recombining. The cut-edges of the rules are shown in dotted lines (Table 8).

The rules in Classes 1–6 generate the graph shown in Fig. 20. Rules in Classes 7–9 introduce q_0 after X and change X into Y to start the simulation in w^R. Rules in Classes 10 and 11 are used to introduce blank B after Y or before Z. Rules in Classes 12–17 simulate the right move of the Turing machine of the form $\delta(q, a) = (q', a', R)$. The subgraph shown in Fig. 21 is changed to the subgraph shown in Fig. 22 by the rules in Class 12. a is removed by the rules in Class 13 and a' is introduced in that place by the rules in Class 14. q is removed by the rules in Class 15 and q' is introduced by the rules in Class 16. Rules in Class 17 remove $[q', a']$. After using the rules in Class 17, we end up with the graph as shown in Fig. 23. Rules in Classes 18 to 23 simulate a left move of the form $\delta(q, a) = (q', a', L)$ in a similar manner. When a final state q_f is reached, rules in Classes 24 to 29 remove from the subgraph as shown

Table 7
Classes of rules in P

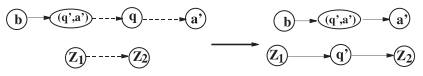
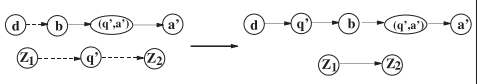
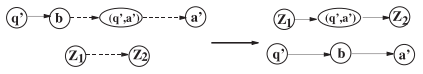
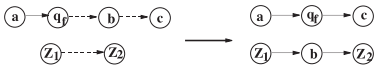
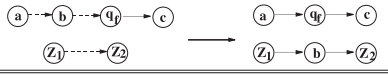
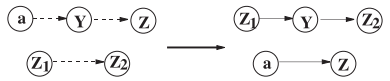
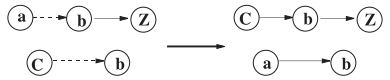
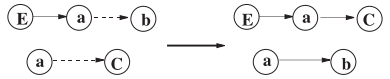
Class 21: In the figure, $b, a' \in \Gamma, q, q' \in Q$. 
Class 22: In the figure, $a', c, b, d \in \Gamma, q' \in Q$. 
Class 23: In the figure, $a', b \in \Gamma, q' \in Q$. 
Class 24: In the figure, $a, c \in N, b \in \Gamma$. 
Class 25: In the figure, $a, c \in N, b \in \Gamma$. 

Table 8
Classes of rules in P

Class 26: 
Class 27: In the figure, $a \in \Sigma$. 
Class 28: In the figure, $a, b \in \Sigma$. 
Class 29: In the figure, $a, b \in \Sigma$. 

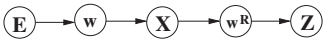


Fig. 20. Graph generated by rules in Classes 1–6.

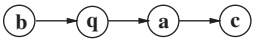


Fig. 21. Subgraph modified by rules in Class 12.

Fig. 22. Subgraph obtained after using the rules in Class 12.

Fig. 23. Graph obtained after using the rules in Class 17.

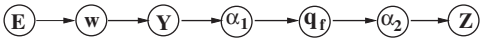


Fig. 24. Graph before applying the rules in Classes 24–29.

in Fig. 24 all other nodes except the nodes representing w between E and Y and we end up with a linear graph representing w.

It can be seen as in the previous section that the graph language generated consists of linear graphs representing strings accepted by the Turing machine M and nothing else. □

6. Conclusions

The paper considers graph splicing systems and proves universality of the model with various restrictions. Graph splicing systems seem to be a good abstract model of systems in which chemical/bio-chemical processes occur. They are also very general in the sense that most formulations of string splicing systems in the literature can be obtained as special cases of graph splicing systems. The result that extended graph splicing systems with only finite sets of axioms and rules are computationally complete, even when some constraints are imposed on them, is a powerful result since most completeness results concerning splicing systems in the literature use either infinite sets of axioms/rules (see [4]) or place artificial constraints on the form of the rules (see [1]).

It is an open problem to find the minimum number of rules required for obtaining computational completeness of the model. It would be worthwhile to modify the definition of the graph splicing systems, for example, to allow for only undirected graphs, and study their generative power.

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